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EQUILIBRIUM STATE OF A CUT ALONG A CIRCUMFERENTIAL ARC IN COMPLEX STRESS STATE*

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Equilibrium state of an arc-like cut in an elastic surface where the crack edges interact with each other in shearlike manner according to Coulomb's Law, is analysed. The nonuniform interaction between the cut edges can lead to formation of zones of adhesion and mutual nonzero displacements of the crack edges. In this case a nonsingular solution is constructed at the boundaries of the zones mentioned, and a method is given for separating such regions.

Problems in which regions of adhesion exist on the interacting cut edges were discussed in /1,2/. The state of stress of an elastic plane with a circumferential arc-like cut without friction at the crack edges was investigated in /3,4/.

In the present paper we consider a cut along a circumference of unit radius. The equation of the cut in the XOY coordinate system has the form (α_0 and β_0 are coordinates of the cut boundaries)

$$x = \cos\theta,\ y = \sin\theta,\ \alpha_0 \leqslant \theta \leqslant \beta_0$$

Mutually perpendicular compressive stresses p, q ($|p| \geqslant |q|$) act at infinity, and the stress p is directed at the angle γ to the OX-axis (Fig.1). The state of stress is described by the complex Kolosov – Muskhelishvili potentials

$$\begin{aligned}
&\sigma_r + \sigma_\theta = 2[\Phi(z) + \Phi^*(z)] \\
&\sigma_r + i\tau_{r\theta} = \Phi(z) + \Omega\left(\frac{1}{z}\right) + z^*\left(z^* - \frac{1}{z}\right)\Psi^*(z) \\
&2G(u' + iv') = iz\left[\varkappa\Phi(z) - \Omega\left(\frac{1}{z}\right) - z^*\left(z - \frac{1}{z}\right)\Psi(z)\right] \\
&\Omega(z) = \Phi^*\left(\frac{1}{z}\right) - \frac{1}{z}\Phi^{*\prime}(z) - \frac{1}{z^2}\Psi^*\left(\frac{1}{z}\right) \\
&(u' = \partial u/\partial\theta',\ v' = \partial V/\partial\theta,\ z = x + iy)
\end{aligned} \tag{1}$$

Here $\varkappa = 3-4\nu$ for the plane deformation, $\varkappa - (3-\nu)/(1-\nu)$ for the generalized plane state of stress, ν is the Poisson's ratio, G is shear modulus and σ_r σ_θ, $\tau_{r\theta}$ are components of the stress tensor in a polar coordinate system with origin at the point O. The displacement components u, v along the OX and OY axes of the rectangular XOY coordinate system are connected with the components v_r and v_θ in the polar coordinate system by

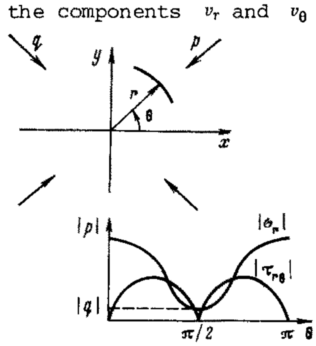

Fig.1 and 2

$$u + iv = (v_r + iv_\theta)e^{i\theta}. \tag{2}$$

The potentials $\Phi(z)$, $\Omega(z)$ must be connected by the following relation /3/:

$$\Phi(0) = \Omega^*(\infty) \tag{3}$$

When $|z| \to \infty$, the complex potentials have the following asymptotics:

$$|z| \to \infty,\ \Phi(z) = \Gamma,\ \Psi(z) = \Gamma',\ \Omega(z) = -\Gamma^{*\prime}/z^2 \tag{4}$$

$$\Gamma = (p + q)/4,\ \Gamma' = -1/2(p - q)e^{-2i\gamma},\ |p| \geqslant |q|$$

We assume that the cut edges interact with each other along the whole crack length ($\sigma_r \leqslant 0$). Mutual displacements may appear at the crack when $\mu|\sigma_r| < |\tau_{r\theta}|$, where σ_r, $\tau_{r\theta}$ are the stresses at the site of the cut in a rigid solid, and μ is the coefficient of friction.

Fig.2 depicts the distribution of the moduli of the normal and shear stresses in a solid without a cut, along the circumference in question. We shall assume that the arc-like cut is situated in a region along which only a single zone of mutual displacement may appear. Such a position of the cut can always be shown using a graph depicting the variation in the normal and shear stresses (e.g. $\pi/2 \leqslant \theta - \gamma \leqslant \pi$ (Fig.2). In the zone of adhesion no mutual displacements appear, therefore we shall regard it as a continuum, and formulate the boundary condition in

*Prikl.Matem.Mekhan.,46,No.3,pp.521-524,1982 the region of mutual displacements. In this case the boundary conditions at the cut will have the form

$$\tau_{r\theta}^{\pm} = \rho\sigma_r^{\pm}, \; r \to R \pm 0, \; \rho = \pm\mu \qquad (5)$$
$$\sigma_r^+ - i\tau_{r\theta}^+ = \sigma_r^- - i\tau_{r\theta}^-, \quad v_r^+ - v_r^- = 0, \quad \alpha \leqslant \theta \leqslant \beta$$

The plus or minus sign preceding μ can be assigned using the direction of the shear stresses appearing at the cut in a rigid solid (e.g. in the case $\pi/2 \leqslant \theta - \gamma \leqslant \pi$ we have $\tau_{r\theta}^{\pm} = \mu\sigma_r^{\pm}$, while for $0 \leqslant \theta - \gamma \leqslant \pi/2$ we have $\tau^{\pm}_{r\theta} = -\mu\sigma_r^{\pm}$, α, β so far unknown regions of mutual displacements).

Using the relations (1), (5) we arrive at the following conjugation problem:

$$(\Phi^+ + \Phi^-)(\rho + i) + (\Phi^{*+} + \Phi^{*-})(\rho - i) + \qquad (6)$$
$$(\Omega^+ + \Omega^-)(\rho + i) + (\Omega^{*+} + \Omega^{*-})(\rho - i) = 0$$
$$\Phi^+ - \Phi^- - (\Omega^+ - \Omega^-) = 0; \; 2G[(u'^+ - u'^-) + i(v'^+ - v'^-)] =$$
$$it\,[\varkappa(\Phi^+ - \Phi^-) + (\Omega^+ - \Omega^-)]$$
$$t = e^{i\theta}, \; \alpha_0 \leqslant \alpha \leqslant \theta \leqslant \beta \leqslant \beta_0$$

We shall seek the solution of the system (6) in the form

$$\Phi(z) = \frac{1}{2\pi}\int_L \frac{\eta(t)\,dt}{t - z} + \Gamma \qquad (7)$$
$$\Omega(z) = \frac{1}{2\pi}\int_L \frac{\eta(t)\,dt}{t - z} + \Gamma - \frac{\Gamma'}{z^2} - D_0, \qquad \alpha \leqslant t \leqslant \beta$$

where we have, with (2), (3) and third equation of (5) taken into account, (L denotes the region of mutual displacements)

$$\eta(\theta) = -\frac{2G}{\varkappa + 1}\left[\frac{u'^+ - u'^- + i(v'^+ - v'^-)}{t}\right] = g - ig'(\theta) \qquad (8)$$
$$g(\theta) = \frac{2G}{\varkappa + 1}(v_\theta^+ - v_\theta^-), \qquad D_0 = \frac{1}{2\pi}\int_L \frac{\eta^*(t)\,dt}{t} = \frac{i}{2\pi}\int_\alpha^\beta g(\theta)\,d\theta$$

The integrals in (7) can be computed at the cut with $z \to t \in L$ using the Sokhotskii—Plemel' formulas /5/. Substituting the expressions (7) for the complex potentials $\Phi(z)$, $\Psi(z)$ we find that two subsequent equations are satisfied indentically, and the first equation is reduced to a singular integral equation the solution of which has the form /5/

$$\eta(t_0)(\rho + i) + \eta^*(t_0)(\rho - i) = -\frac{1}{\pi x(t_0)}\int_L \frac{x(t)\,p(t)}{t - t_0}\,dt + \frac{Ci}{x(t_0)}, \qquad t_0 \in L \qquad (9)$$
$$C = (1 - i\rho)\int_L \eta^*(t)\,dt; \qquad p(t) = 2D_0\rho + \frac{\Gamma^{*\prime}(\rho + i)}{t_0^2} + \Gamma' t_0^2(\rho - i) - 4\Gamma\rho$$
$$x(z) = \sqrt{(z - a)(z - b)}, \qquad a = e^{i\alpha}, \qquad b = e^{i\beta};$$
$$|z| \to +\infty, \qquad x(z) = z - \frac{a + b}{2}z$$

Computing the integral in the right-hand side of (9) we obtain

$$\eta(t)(\rho + i) + \eta^*(t)(\rho - i) = \frac{i}{x(t)}(A(t) + C), \qquad t \in L \qquad (10)$$
$$A(t) = A_1t^3 + A_2t^2 + A_3t + A_4 + \frac{D_1}{t} + \frac{D_2}{t^2}$$
$$A_1 = \Gamma'(\rho - i), \quad A_2 = \frac{a + b}{2}\Gamma'(\rho - i), \quad A_3 = -\frac{(a - b)^2}{8}\Gamma'(\rho - i) + B$$
$$A_4 = -\frac{(a + b)(a - b)^2}{16}A_1 - B\left(\frac{a + b}{2}\right)$$
$$B = 2D_0\rho - 4\Gamma\rho, \qquad D_1 = -\frac{a + b}{2\sqrt{ab}}\Gamma^{*\prime}(\rho + i)$$
$$D_2 = -\Gamma^{*\prime}(\rho + i)\sqrt{ab} = -A_1^*\sqrt{ab}$$

Equation (10) contains two unknown constants, C and D_0. Multiplying both sides of (10) by t^{-1} and integrating along the cut L, we obtain

$$C = -A_4 - A_3^*\sqrt{ab} \qquad (11)$$

Substituting now the expression for the function η (8) and using (11), we arrive at a differential equation the solution of which has the form

$$g(\theta) = \left[F_1t + F_2 + \frac{F_3}{t} + \frac{F_4}{t^2}\right]\frac{x(t)}{2} + \frac{R(t)}{2} \qquad (12)$$

$$R(t) = e^{-\rho\theta} \int_{\alpha}^{\theta} \frac{K_1 t + K_2}{x(t)} e^{\rho\theta} \, d\theta$$

$$F_1 = \frac{iA_1}{\rho + 2i}, \quad F_2 = \frac{F_1(a+b)}{2}, \quad F_3 = \frac{F_2^*}{\sqrt{ab}}, \quad F_4 = \frac{F_1^*}{\sqrt{ab}}$$

$$K_1 = iA_3 - abF_1(\rho + i) + F_2(a+b)(\rho + {}^1/_2 i) - F_3\rho, \quad K_2 = K_1^* \sqrt{ab}$$

The condition that the displacements are zero at the cut ends and using (12), together yield the constant D_0. We have

$$D_0 = i\frac{MC_1 + M^* C_1^*}{2\rho(C_1 + C_1^*)}, \qquad C_1 = \int_{\alpha}^{\beta} \frac{te^{\rho\theta}}{x(t)} d\theta, \qquad t \in L \tag{13}$$

$$M = -\frac{i(a-b)^2 A_1}{8} - 4\Gamma\rho i + K_1 - iA_3$$

In the general case the expressions (13), (12) and (8) determine the singular solutions at the points $t = a$, $t = b$ characterized by the intensity coefficients /4,6/

$$K^- = \lim_{\theta \to \alpha} \sqrt{2(\theta - \alpha)}\, g'(\theta), \qquad K^+ = \lim_{\theta \to \beta} \sqrt{2(\beta - (\theta)}\, g'(\theta); \; \alpha \leqslant \theta \leqslant a$$

The singular shear stresses at the cut near the boundary of the region of mutual displacements will always cause the coupling of the clamped segment of the crack, therefore we shall construct a nonsingular solution at the boundary of the adhesion and mutual displacement zone. Following the method given in /7,8/, we shall regard, as a criterion of equilibrium, the equality of the stress intensity coefficients to zero

$$C + A(t) = 0 \tag{14}$$

Equation (14) determines the unknown boundaries of the adhesion and mutual displacement zone. If both roots of the equation lie within the cut $\alpha_0 \leqslant \theta \leqslant \beta_0$, then the adhesion zone is adjacent to the right and left tip of the initial cut. If on the other hand one of the roots lies outside the cut, then the regions of adhesion which appears is adjacent to one boundary of the cut, and the zone of mutual displacement to the other boundary. Thus the equation (14) and (13), (12), (8), (7) together determine the state of stress in a solid weakened by a plane, arc-like crack.

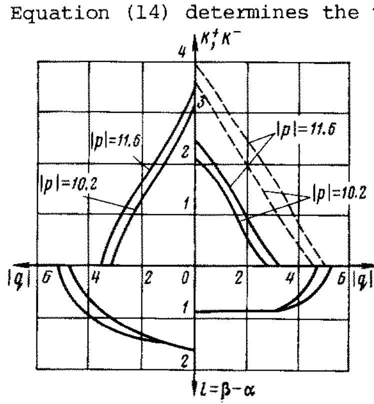

Fig.3

An an example we shall consider a cut situated in the region $\pi/2 \leqslant \theta \leqslant \pi$; $\mu = 0.4$; $\gamma = 0$. In this case we have in the expression (5) $\rho = +\mu$. The condition of crack growth has the form /4/ $K^{\pm} \geqslant K_c$ where K_c denotes the coupling modulus. Fig.3 depicts a graph showing the dependence of the intensity coefficients on the stresses p, q (K^- corresponds to the solid curve and K^+ to the dashed curve) and the length of displacement zone. The curves on the left and right correspond to the cut parameters $\alpha_0 = \pi/2$, $\beta_0 = \pi$ and $-\alpha_0 = \pi/2$, $\beta_0 = 3\pi/4$ respectively. We see here that the intensity coefficients increase with increasing shear stress, and this may lead to cracks propagation even in the presence of a zone of adhesion.

The proposed method of determining exact boundaries of the adhesion and mutual displacement zones can be generalized to include the construction of a nonsingular solution for rectilinear and curvilinear cuts in a complex stress state.

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